CCRT: Categorical and Combinatorial Representation Theory.

From combinatorics of universal problems
to usual applications.

G.H.E. Duchamp
Collaboration at various stages of the work
and in the framework of the Project
Evolution Equations in Combinatorics and Physics :
Karol A. Penson, Darij Grinberg, Hoang Ngoc Minh, C. Lavault,
C. Tollu, N. Behr, V. Dinh, C. Bui,
Q.H. Ngô, N. Gargava, S. Goodenough.

CIP seminar,

Friday conversations:

For this seminar, please have a look at Slide CCRT[n] & ff.

Goal of this series of talks

The goal of these talks is threefold

1. Category theory aimed at "free formulas" and their combinatorics
2. How to construct free objects
   1. w.r.t. a functor with - at least - two combinatorial applications:
      1. the two routes to reach the free algebra
      2. alphabets interpolating between commutative and non commutative worlds
   2. without functor: sums, tensor and free products
   3. w.r.t. a diagram: limits
3. Representation theory: Categories of modules, semi-simplicity, isomorphism classes i.e. the framework of Kronecker coefficients.
4. MRS factorisation: A local system of coordinates for Hausdorff groups.

CCRT[16] Higher order BTT (part 1).

Disclaimer. – The contents of these notes are by no means intended to be a complete theory. Rather, they outline the start of a program of work which has still not been carried out.

1. Paths drawn on the Magnus group.
2. Local analysis
3. Integration and Picard's process
4. Analysis of the classical BTT
5. todo: from now Definition of evolution equations
6. Computations with differential modules
7. Some concluding remarks

Introduction

1. Today, and in subsequent parts, we are going to analyse the first form of the BTT and its extensions (Proof, Boundaries investigations (Bourbaki's method), Picard's generalized process, closed subgroups, localization, &c.).
2. The mental process for the making of the BTT [9] will be the following

$$\underbrace{\text{Slicing the obervable} \rightarrow \ \text{Differentiation} \rightarrow}_{\textit{Analysis}}$$

$$\underbrace{\text{Technical condition} \rightarrow \ \text{NSC} \rightarrow \ \text{Proof \& Boundaries}}_{\textit{Synthesis / Integration}}$$

3. This method is not new, it is that of Archimedes (-287, -212) [1], Liu Hui (220-280) [13] and Cavalieri (1598-1647) [6]. Archimedes work was originally thought to be lost, but in 1906 was rediscovered in the celebrated Archimedes Palimpsest.

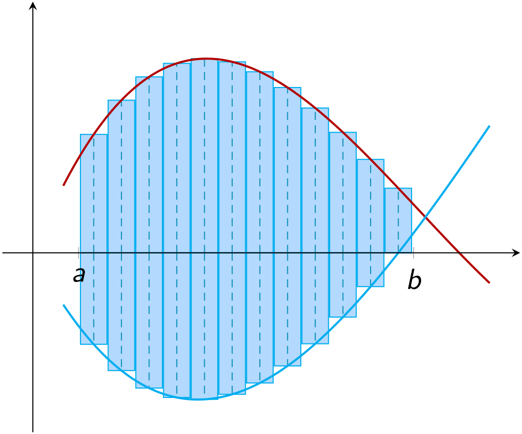


Figure: Analysing the observable:
Here sampling.

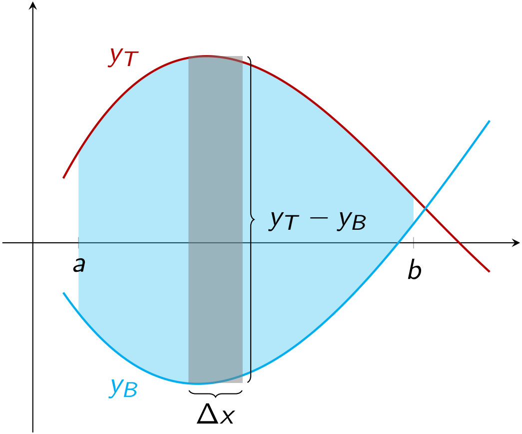


Figure: Analysing the observable:
Limiting process.
Synthesising: integration.

Remarks

1. Samplings

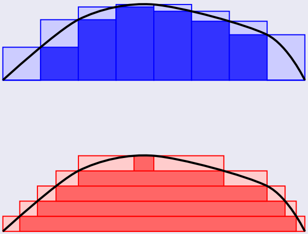
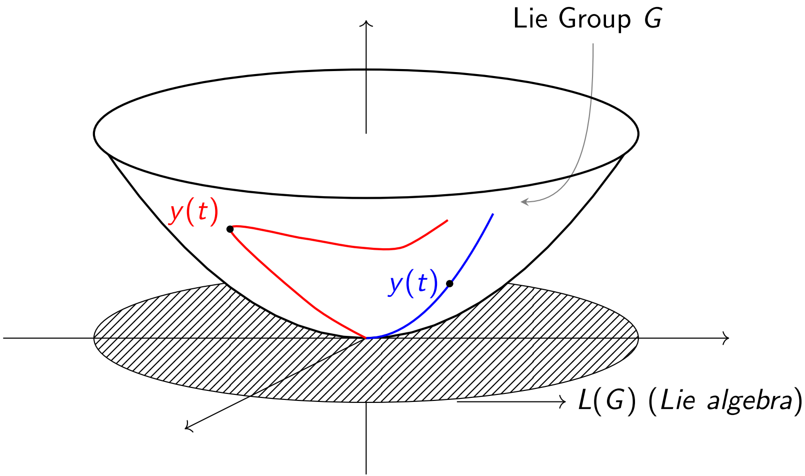


Figure: Lebesgue's integral is based on another concept (y-axis sampling and measurement of measurable sets). Its merit is to be compatible with pointwise limits.

2. Riemann integral being based on x-axis sampling is connected to antiderivatives and differential algebra.

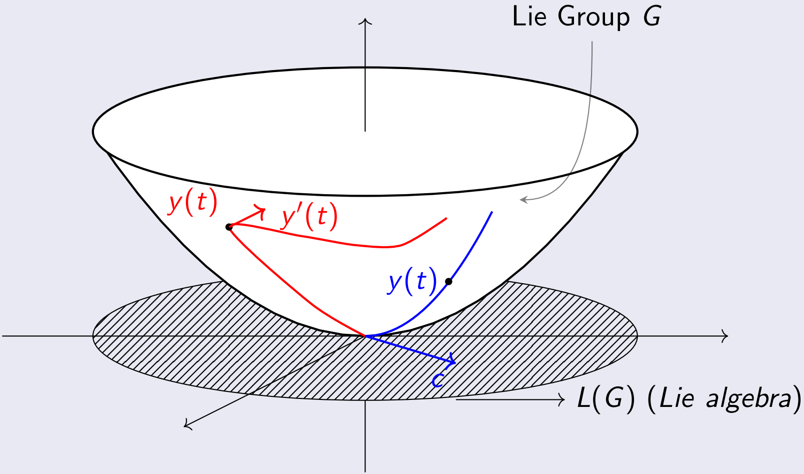
Paths drawn on Magnus groups.

4. We start from the picture of CCRT[10] (with two paths drawn)



Towards BTT: paths drawn on Magnus groups./2

5. In this case $y'(t)y(t)^{-1} = m(t)$ is a path drawn on the tengent space of $1 + \mathbb{C}_+\langle\langle X\rangle\rangle$ i.e. drawn on $\mathbb{C}_+\langle\langle X\rangle\rangle$ which amounts, for $t = z \in \Omega$ to $M \in \mathcal{H}(\Omega)_+\langle\langle X\rangle\rangle$.



Paths drawn on Magnus groups./3

6. Conversely, given any $M \in \mathcal{H}(\Omega)_+\langle\langle X\rangle\rangle$, the system

$$\left\{ \begin{array}{rcl} \mathbf{d}(S) & = & M.S \\ S(z_0) & = & 1_{\mathcal{H}(\Omega)\langle\langle X\rangle\rangle} \end{array} \right. \tag{1}$$

has a unique solution which is the limit of Picard's process

$$S_0 = 1_{X^*} \ ; \ S_{n+1} = 1_{X^*} + \int_{z_0}^{z} M.S_n(s)ds \tag{2}$$

7. This limit is stationary due to

$$S_n = S_0 + \sum_{0\le j\le n-1} S_{j+1} - S_j \text{ and } S_{j+2} - S_{j+1} = \int_{z_0}^{z} M.(S_{j+1} - S_j)$$

Paths drawn on Magnus groups./4

8. There are many ways to obtain regular paths drawn on Magnus groups. For example, setting $\mathbb{L} = \sum_{w\in X^*} \mathrm{Li}_w \; w$, the series $T = \mathbb{L}e^{-x_0 \log(z)}$ satifies

$$\left\{ \begin{array}{rcl} \mathbf{d}(T) & = & (\frac{x_0}{z} + \frac{x_1}{1-z}).T + T(\frac{-x_0}{z}) \\ \lim_{z\to 0} T(z) & = & 1_{\mathcal{H}(\Omega)\langle\langle X\rangle\rangle} \end{array} \right. \tag{3}$$

9. Such systems can always be converted to systems with left multipliers with $M(z) = \mathbf{d}(S)S^{-1}$

What is BTT ?

10. **What is BTT ?. – (Concrete form)** BTT concerns "step-one" iterated integrals. The framework is the following
Let $\Omega \subset \mathbb{C}$ (a connected open subset) and a family of inputs $(u_x)_{x\in X}$ where $u_x \in \mathcal{C}$ ($\mathcal{C}$ is a differential subfield of $\mathcal{M}(\Omega) = Frac(\mathcal{H}(\Omega))$ containing $\mathbb{C}$)
Concrete BTT is a theorem which gives necessary and sufficient conditions for all the tree of coordinates (i.e. the family $(\langle S|w\rangle)_{w\in X^*}$ of a solution of a system (4) to be $\mathcal{C}$ linearly independent). More precisely, given a family of inputs $(u_x)_{x\in X}$ we suppose that S satisfies

$$\begin{cases} \mathbf{d}(S) &= M.S \text{ with } M = \sum_{x\in X} u_x\, x \\ \langle S|1_{X^*}\rangle &= 1_{\mathcal{H}(\Omega)\langle\langle X\rangle\rangle} \end{cases} \tag{4}$$

BTT gives criteria for $(\langle S|w\rangle)_{w\in X^*}$ to be $\mathcal{C}$-linearly free.

What is BTT ?/2

11. **Why BTT abstract form ?. –** BTT has been cited 48 times so far (30/03/21). Outside of our school it has been used as a crucial point at least 2 times [14, 16].
    For the following reasons, we must have a general, characteristic-free form (and its variations).
    1. It provides deep reasons, is easy to analyse and create variations
    2. It implies the concrete form and is a good test for boundaries (conditions and function field/algebra)
    3. It can be reused in other contexts [14, 16]

12. **What is BTT abstract form ?. –**
    1. The algebra $(\mathcal{H}(\Omega), d/dz)$ is replaced differential algebra $(\mathcal{A}, \partial)$ (not necessarily graded, and not necessarily sectioned).
    2. Inputs belong to a differential subfield $\mathcal{C}$ containing the constants.
    3. The system $\mathbf{d}(S) = M.S,\ \langle S|1_{X^*}\rangle = 1_{\mathcal{A}}$ is considered *as already integrated* with various multipliers.

BTT original: hypotheses.

Theorem (DDMS Original, and characteristic-free, [9])

Let $(\mathcal{A},\partial)$ *be a commutative differential ring and* $\mathcal{C}$ *a differential subfield of* $\mathcal{A}$ *containing the constants* $\mathbf{k}=\ker(\partial)$ *(i.e.* $\partial(\mathcal{C})\subset\mathcal{C}\supset\mathbf{k}$*). We suppose that* $S\in\mathcal{A}\langle\langle X\rangle\rangle$ *is a solution of the differential equation*

$$\mathbf{d}(S)=M.S\ ;\ \langle S|1_{X^*}\rangle=1_{\mathcal{A}} \tag{5}$$

where the multiplier M *is an homogeneous series (a polynomial in the case of finite* X*) of degree* 1*, i.e.*

$$M=\sum_{x\in X}u_x\,x\in\mathcal{C}\langle\langle X\rangle\rangle\ . \tag{6}$$

and the differential operator acts termwise on series i.e.

$$\mathbf{d}(S)=\textstyle\sum_{w\in X^*}\partial(\langle S|w\rangle)\,w$$

.

BTT original: chain of equivalent conditions

Theorem (cont'd)

The following conditions are equivalent :

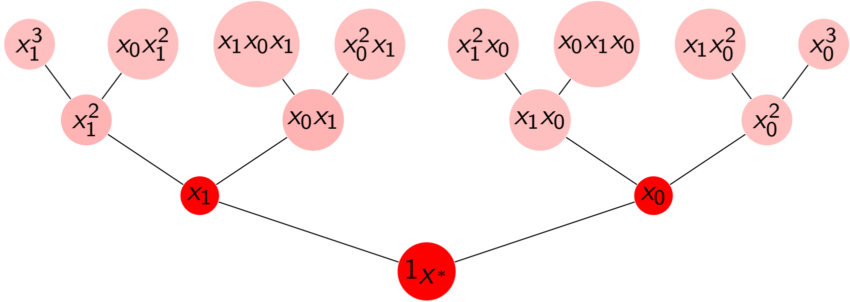
1. *The family* $(\langle S|w\rangle)_{w\in X^*}$ *of coefficients of* S *is free over* $\mathcal{C}$.
2. *The family of coefficients* $(\langle S|y\rangle)_{y\in X\cup\{1_{X^*}\}}$ *is free over* $\mathcal{C}$.
3. *The family* $(u_x)_{x\in X}$ *is such that, for* $f\in\mathcal{C}$ *and* $\alpha_x\in\mathbf{k}$

$$\partial(f)=\sum_{x\in X}\alpha_x u_x \Longrightarrow (\forall x\in X)(\alpha_x=0)\ . \tag{7}$$

4. *The family* $(u_x)_{x\in X}$ *is* $\mathbf{k}$*-free and*

$$\partial(\mathcal{C})\cap span_{\mathbf{k}}\Big((u_x)_{x\in X}\Big)=\{0\}\ . \tag{8}$$

Illustration with iterated integrals



Some coefficients with $X = \{x_0, x_1\}$; $u_0(z) = \frac{1}{z}$; $u_1(z) = \frac{1}{1-z}$, $*_0 = 0$

$$\langle S|x_1^n\rangle = \frac{(-log(1-z))^n}{n!} \quad ; \quad \langle S|x_0x_1\rangle = \underbrace{\mathrm{Li}_2(z)}_{cl.not.} = \mathrm{Li}_{x_0x_1}(z) = \sum_{n\geq 1}\frac{z^n}{n^2}$$

$$\langle S|x_0^2x_1\rangle = \underbrace{\mathrm{Li}_3(z)}_{cl.not.} = \mathrm{Li}_{x_0^2x_1}(z) = \sum_{n\geq 1}\frac{z^n}{n^3} \quad ; \quad \langle S|x_1x_0x_1\rangle = \mathrm{Li}_{x_1x_0x_1}(z) = \mathrm{Li}_{[1,2]}(z) = \sum_{n_1>n_2\geq 1}\frac{z^{n_1}}{n_1n_2^2}$$

$$\langle S|x_0x_1^2\rangle = \mathrm{Li}_{x_0x_1^2}(z) = \mathrm{Li}_{[2,1]}(z) = \sum_{n_1>n_2\geq 1}\frac{z^{n_1}}{n_1^2n_2}$$

; above "cl. not." stands for "classical notation"

Insights from the proof of BTT

i $\Longrightarrow$ ii Obvious, by restriction. Condition ii is that of the "basic triangle" (we will see that it is robust to localisation)

ii $\Longrightarrow$ iii This condition comes from the consideration of fuchsian multipliers (i.e. $M = \sum_{x \in X} \frac{\lambda_x x}{z - a_x}$, a_x all different and $\lambda_x \neq 0$), as if $f \in \mathcal{C} = \mathbb{C}(z)$, the derivative $\partial(f)$ has no simple pole.

iii $\Longleftrightarrow$ iv This condition (iv) is simply a reformulation of (iii) in terms of subspaces.

Proof of BTT itself/1

13. The first implications

    (i) $\Longrightarrow$ (ii) Obvious, by restriction.

    (ii) $\Longrightarrow$ (iii) This condition comes from the consideration of fuchsian multipliers (i.e. $M = \sum_{x \in X} \frac{\lambda_x\, x}{z - a_x}$, a_x all different and $\lambda_x \neq 0$), as if $f \in \mathcal{C} = \mathbb{C}(z)$, the derivative $\partial(f)$ has no simple pole.
    At the logical level (and in full generality), if one has a relation as in the LHS of (7), then $f - \sum_{x \in X} \alpha_x \langle S|x \rangle$ must be a constant c and then $(f - c).1_{\mathcal{A}} - \sum_{x \in X} \alpha_x \langle S|x \rangle = 0$ and from (ii), one gets $(\alpha_x)_{x \in X} = 0$.

    (iii) $\Longleftrightarrow$ (iv) This condition (iv) is simply a reformulation of (iii) in terms of subspaces.

Proof of BTT itself/2: the hard part.

14. The hardest part
    iii $\Longleftrightarrow$ i This is the hard (and most mysterious) part.
    Remark that, due to the pairing

$$\mathcal{A}\langle\langle X\rangle\rangle \otimes_{\mathcal{C}} \mathcal{C}\langle X\rangle \to \mathcal{A} \tag{9}$$

    the module $\mathcal{R}$ of $\mathcal{C}$-linear relations between the family of coefficients $(\langle S|w\rangle)_{w\in X^*}$ is the space ($\mathcal{C}$-submodule, actually)

$$\mathcal{R} = \ker(P \mapsto \langle S|P\rangle) \tag{10}$$

    of polynomials $\mathcal{C}\langle X\rangle$ such that $\langle S|P\rangle = 0_{\mathcal{A}}$.
15. If $\mathcal{R} = \{0_{\mathcal{C}\langle X\rangle}\}$, we are done (i.e. $(\langle S|w\rangle_{w\in X^*}$ is $\mathcal{C}$-free). Let us now deal with the other case ($\mathcal{R} \neq \{0_{\mathcal{C}\langle X\rangle}\}$).

Proof of BTT itself/3

16. In this last case, we take a well ordering $\prec_{wo}$ on X and order words by $\prec_{grlex}$ i.e.

$$u \prec_{grlex} v \iff |u| < |v| \text{ or } u = pxs_1 \ v = pys_2 \text{ and } x \prec_{wo} y$$

Every non zero $Q \in \mathcal{R} \setminus \{0\}$ reads

$$Q = \lambda.w + \sum_{u \prec_{grlex} w} \langle P|u\rangle\, u = \lambda.lead(Q) + \sum_{u \prec_{grlex} w} \langle P|u\rangle\, u \tag{11}$$

We choose P as the $Q \in \mathcal{R} \setminus \{0\}$ with the least $lead(P)$ and, **due to the fact that $\mathcal{C}$ is a field**, we can suppose P to be monic, $\langle P|lead(P)\rangle = 1$.

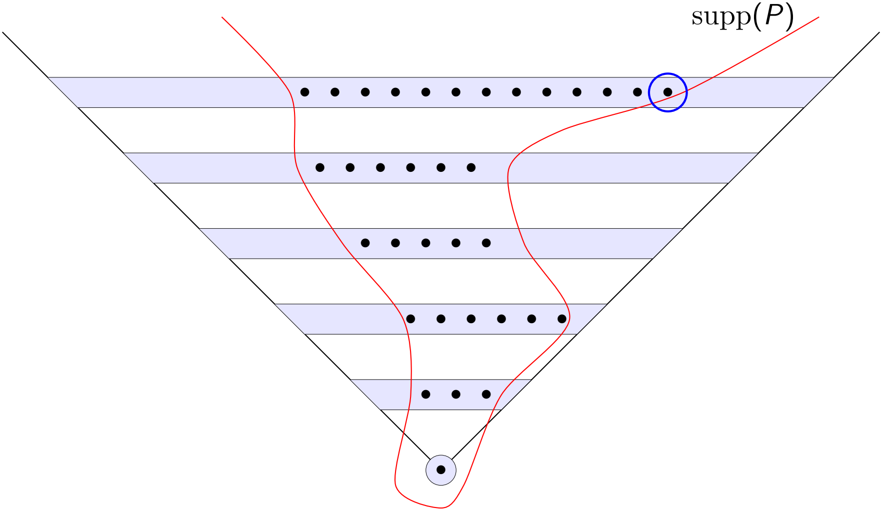


Figure: A polynomial $P \in \mathcal{C}\langle X \rangle \setminus \{0\}$, its support is drawn with black spots. The homogeneous slices are designed in pale blue. The leader monomial *lead*(P) (within the blue circle) is the rightmost word of the support in the upper row.

Proof of BTT itself/4

17. Now comes two important remarks
    1. Pairing (9) follows Leibniz rule i.e.

$$\partial(\langle S|P\rangle) = \langle \mathbf{d}(S)|P\rangle + \langle S|\mathbf{d}(P)\rangle$$

    2. We can transpose multiplication (one of the greatest tricks in mathematics: Cayley and Lagrange thms, Ring theory, Harmonic analysis, Distribution theory, Symmetric functions, shifts in Spectral theory).
18. For that particular polynomial (minimal monic in $\mathcal{R} \setminus \{0_{\mathcal{C}\langle X\rangle}\}$), we compute

$$\begin{aligned}\partial(\langle S|P\rangle) &= \langle MS|P\rangle + \langle S|P'\rangle = \langle S|M^{-1}P\rangle + \langle S|P'\rangle \\ &= \langle S|M^{-1}P + P'\rangle \end{aligned} \tag{12}$$

Proof of BTT itself/5

19. We have the following lemma

Lemma

Let R be a ring, X an alphabet, $T, A \in R\langle\langle X\rangle\rangle, B \in R\langle X\rangle$, then
i) We have

$$\langle TA|B\rangle = \langle A|T^{-1}B\rangle \tag{13}$$

where $T^{-1}B \in R\langle X\rangle$ is such that, for all $w \in X^*$, $\langle T^{-1}B|w\rangle = \langle B|Tw\rangle$.
ii) If R_1 is a subring and $T \in R_1\langle\langle X\rangle\rangle$ (resp. $B \in R_1\langle X\rangle$) then $T^{-1}B \in R_1\langle X\rangle$.

Proof.

Left to the reader. □

Proof of BTT itself/6

20. Restarting from (12), we get $M^{-1}P + P' \in \mathcal{R}$ which cannot be $\neq 0$ because, otherwise, we would have
$$lead(M^{-1}P + P') \prec_{grlex} lead(P)$$
21. Then $P' = -M^{-1}P$
22. Firstly, for $|w| = |lead(P)|$ (the uppermost row), we have
$$\partial(\langle P|w\rangle) = \langle P'|w\rangle = -\sum_{x\in X} u_x \langle P|xw\rangle = 0$$
then, for $|w| = |lead(P)|$, $\langle P|w\rangle \in \ker(\partial) = \mathbf{k}$
23. Secondly, as it is impossible that $lead(P) = 1_{X^*}$, we can set $lead(P) = yu$ ($y \in X$, initial letter) and likewise
$$\partial\Big(\underbrace{\langle P|u\rangle}_{\in\mathcal{C}} \Big) = -\sum_{x\in X} u_x \underbrace{\langle P|xu\rangle}_{\in\mathbf{k}} = 0$$
we are then in the configuration of condition (iii) in BTT, then all coefficients $(\langle P|xu\rangle)_{x\in X}$ are zero including $\langle P|yu\rangle$, contrariwise to the initial normalisation $\langle P|lead(P)\rangle = 1$.

Localization of the BTT

Theorem (DGMSV [10])

Let $(\mathcal{A},\partial)$ be a commutative associative differential ring with subring of constants $\mathbf{k}=\ker(\partial)$. Let $\mathcal{C}$ be a differential subring (i.e. $\partial(\mathcal{C})\subset\mathcal{C}$) of $\mathcal{A}$ which is an integral domain such that $\ker(\partial_{Frac(\mathcal{C})})=Frac(\mathbf{k})$.
We suppose that, for all $x\in X$, $u_x\in\mathcal{C}$ and that $S\in\mathcal{A}\langle\langle X\rangle\rangle$ is a solution of the differential equation (5).
Then TFAE:

1. *The family $(\langle S|w\rangle)_{w\in X^*}$ of coefficients of S is free over $\mathcal{C}$.*
2. *The family of coefficients $(\langle S|y\rangle)_{y\in X\cup\{1_{X^*}\}}$ is free over $\mathcal{C}$.*
3. *For all $f_1,f_2\in\mathcal{C}$, $f_2\neq 0$ and $\alpha\in k^{(X)}$, we have the property*
$$W(f_2,f_1)=f_2^2(\sum_{x\in X}\alpha_x u_x)\Longrightarrow(\forall x\in X)(\alpha_x=0)\ . \qquad (14)$$
where $W(f_2,f_1)$, the wronskian, stands for $\partial(f_1)f_2-f_1\partial(f_2)$.

Discussion

i). – We have supposed that $\ker(\partial_{Frac(\mathcal{C})}) = Frac(\mathbf{k})$ because it is not granted that this holds from $\mathbf{k} \subset \mathcal{C}$. Indeed, take $\mathcal{A} = \mathcal{C} = \mathbb{C}[x, y]$ and $D = x\partial_x + y\partial_y$ (Euler total degree number operator, already met, see CCRT[15], slide 12 and ff). Then $\mathbf{k} = \mathbb{C}$ and nevertheless $x/y \in \ker(\partial_{Frac(\mathcal{C})})$. See also discussion in [22].
ii). – In fact, in the localized form and with $\mathcal{C}$ not a differential field, condition (iii) (slide 15) is strictly weaker than (iii) (slide 25), as shows the following family of counterexamples

1. $\Omega = \mathbb{C} \setminus (]-\infty, 0])$
2. $X = \{x_0\}$, $u_0 = z^\beta$, $\beta \notin \mathbb{Q}$
3. $\mathcal{C}_0 = \mathbb{C}\{\{z^\beta\}\} = \mathbb{C}.1_\Omega \oplus span_{\mathbb{C}}\{z^{(k+1)\beta - l}\}_{k,l \geq 0}$
4. $S = 1_\Omega + (\sum_{n\geq 1} \frac{z^{n(\beta+1)}}{(\beta+1)^n n!})$

Let us show that, for these data (iii) holds but not (i).
Firstly, we show that $\mathcal{C}_0 = \mathbb{C}\{\{z^\beta\}\}$ corresponds to the given direct sum. We remark that the family $(z^\alpha)_{\alpha\in\mathbb{C}}$ is $\mathbb{C}$-linearly free (within $\mathcal{H}(\Omega)$), which is a consequence of the fact that they are eigenfunctions, for different eigenvalues, of the Euler operator $z\frac{d}{dz}$.

Then

$$\mathbb{C}\{\{z^{\beta}\}\} = \mathbb{C}1_{\Omega} \oplus span_{\mathbb{C}}\{z^{(k+1)\beta-l}\}_{k,l\geq 0} = span_{\mathbb{C}}\{z^{(k')\beta-l}\}_{k',l\geq 0}$$

comes from the fact that the RHS is a subset of the LHS as, for all, $k, l \geq 0$, $z^{(k+1)\beta-l} \in \mathbb{C}\{\{z^{\beta}\}\}$. Finally $1_{\Omega} \in \mathbb{C}\{\{z^{\beta}\}\}$ by definition ($\mathbb{C}\{\{X\}\}$ is a $\mathbb{C}$-AAU).

(iii) is fulfilled. Here

$u_0(z) = z^{\beta}$ is such that, for any $f \in \mathcal{C}_0$ and c_0 in $\mathbb{C}$, we have

$$c_0 u_0 = \partial_z(f) \Longrightarrow (c_0 = 0) \tag{15}$$

But (i) is not Because we have the following relation

$$(\beta+1)z^{\beta-1}\langle S|x_0\rangle - z^{2\beta}.1_{\Omega} = 0$$

Sketch of the proof

After some technicalities, we show that (5) can be transported in $\mathcal{A}[(\mathcal{C}^{\times})^{-1}]$ by means of the following commutative diagram and back.

$$
\begin{tikzcd}
\mathcal{C} \arrow[rr, hook, "\varphi_{\mathcal{C}}"] \arrow[dr, hook, "j"'] \arrow[dd, "d"'] & & Fr(\mathcal{C}) \arrow[dr, hook, "j_{frac}"] \arrow[dd, "d_{frac}" near start] & \\
& \mathcal{A} \arrow[rr, "\varphi_{\mathcal{A}}" near start] \arrow[dd, "d" near start] & & \mathcal{A}[(\mathcal{C}^{\times})^{-1}] \arrow[dd, "d_{frac}"] \\
\mathcal{C} \arrow[rr, hook, "\varphi_{\mathcal{C}}" near end] \arrow[dr, hook, "j"'] & & Fr(\mathcal{C}) \arrow[dr, hook, "j_{frac}"'] & \\
& \mathcal{A} \arrow[rr, "\varphi_{\mathcal{A}}"] & & \mathcal{A}[(\mathcal{C}^{\times})^{-1}]
\end{tikzcd}
\tag{16}
$$

Proof that $[1_\Omega, \log(z), \log(\frac{1}{1-z})]$ is $\mathcal{C}_\mathbb{R}$-free.

Recall that $\mathcal{C}_\mathbb{R} = \mathbb{C}[(z^\alpha, (1-z)^{-\beta})_{\alpha,\beta\in\mathbb{R}}]$ and let us suppose $P_i, i = 1\ldots 3$ such that

$$P_1(z) + P_2(z)\log(z) + P_3(z)(\log(\frac{1}{1-z})) = 0_\Omega$$

We first prove that $P_2 = \sum_{i\in F} c_i z^{\alpha_i}(1-z)^{\beta_i}$ is zero using the deck transformation D_0 of index one around zero.
One has $D_0^n(\sum_{i\in F} c_i z^{\alpha_i}(1-z)^{\beta_i}) = \sum_{i\in F} c_i z^{\alpha_i}(1-z)^{\beta_i} e^{2i\pi.n\alpha_i}$, the same calculation holds for all P_i which proves that all $D_0^n(P_i)$ are bounded. But one has $D_0^n(\log(z)) = \log(z) + 2i\pi.n$ and then

$$D_0^n(P_1(z) + P_2(z)\log(z) + P_3(z)(\log(\frac{1}{1-z}))) =$$
$$D_0^n(P_1(z)) + D_0^n(P_2(z))(\log(z) + 2i\pi.n) + D_0^n(P_3(z))\log(\frac{1}{1-z}) = 0$$

It suffices to build a sequence of integers $n_j \to +\infty$ such that $\lim_{j\to\infty} D_0^{n_j}(P_2(z)) = P_2(z)$ which is a consequence of the following lemma.

Lemma

Let us consider a homomorphism $\varphi : \mathbb{N} \to G$ where G is a compact (Hausdorff) group, then it exists $u_j \to +\infty$ such that

$$\lim_{j\to\infty} \varphi(u_j) = e$$

Proof.

First of all, due to the compactness of G, the sequence $\varphi(n)$ admits a subsequence $\varphi(n_k)$ convergent to some $\ell \in G$. Now one can refine the sequence as n_{k_j} such that

$$0 < n_{k_1} - n_{k_0} < \ldots < n_{k_{j+1}} - n_{k_j} < n_{k_{j+2}} - n_{k_{j+1}} < \ldots$$

With $u_j = n_{k_{j+1}} - n_{k_j}$ one has $\lim_{j\to\infty} \varphi(u_j) = e$.
End of the proof One applies the lemma to the morphism

$$n \mapsto (e^{2i\pi . n\alpha_i})_{i\in F} \in \mathbb{U}^F$$

Conclusion

- For Series with variable coefficients, we have a theory of Noncommutative Evolution Equation sufficiently powerful to cover iterated integrals and multiplicative renormalisation
- Use of combinatorics on words gives a necessary and sufficient condition on the "inputs" to have linear independance of the solutions over higher function fields.
- Picard (Chen) solutions admit enlarged indexing w.r.t. compact convergence on Ω (polylogarithmic case) but Drinfeld's G_0 has a domain which includes only some rational series.
- Localization is possible (under certain conditions).
- Local BTT theorem allows to explore linear and algebraic independences w.r.t. subalgebras of $Dom(\mathrm{Li})$.

THANK YOU FOR YOUR ATTENTION !

[1] Archimedes method of Mechanical Theorems https://en.wikipedia.org/wiki/The_Method_of_Mechanical_Theorems

[2] J. Berstel and C. Reutenauer, *Noncommutative Rational Series with Applications*, Cambridge University Press, March 2013 (Online), 2009 (Print).

[3] N. Bourbaki, *Theory of sets*, Springer-Verlag Berlin Heidelberg 2004

[4] N. Bourbaki.– *Algebra ch 1-3*, Springer-Verlag Berlin and Heidelberg GmbH & Co. K; (2nd printing 1989)

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